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# An accurate beam theory and its first-order approximation

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**Abstract**

An infinite system of one-dimensional differential equations is derived from the two-dimensional theory of elasticity by expanding the displacement field in a series of trigonometrical functions together with a linear term. Since the trigonometrical functions are pure-thickness-vibration modes of infinite plates or beams with the top and bottom surfaces being free, the differential equations, and the corresponding boundary conditions lay a basis of an accurate beam theory for natural vibration analysis of beams. A high-order approximation of the infinite system should be quite useful on the analysis of beams at high frequencies. This paper pays a special attention on its first-order approximation, which leads to a first-order shear deformation beam theory for flexural vibrations. The differential equations in the first-order approximation of present beam theory can be transformed to those in Timoshenko's beam theory. The most important difference between Timoshenko's beam theory and the present first-order shear deformation beam theory is the different field displacements. For the assessment of the accuracy of first-order approxmation of present beam theory, the numerical results of frequencies of natural vibrations, frequency spectra, and mode shapes of beams with classical boundary conditions by the first-order approximation of present beam theory are compared with those by Timoshenko's beam theory and plane stress theory. Considering the plane stress theory as an reference, the first-order approximation of present beam theory is a little more accurate at describing the field shapes of beams than Timoshenko's beam theory. Therefore, the present beam theory is an addition to the existing beam theory with improved accuracy for the vibration analysis of beams.

*Keywords:* Vibration, Beam, Shear deformation, Frequency

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## 1. Introduction

The vibration analysis of beams plays a very important role in the design of structures in many industries including aerospace, automobile, ship-building, construction, amongst others. From a historical perspective, the most significant theoretical development of beam was by Euler and Bernoulli, about 250 years ago. Much later in the early twentieth century, it was improved by Timoshenko [1]. The Euler-Bernoulli beam is applicable for slender beams, while the Timoshenko beam is applicable for thick or deep beams, since it considers the shear deformation and rotatory inertia effects. In Timoshenko's beam theory, the transverse shear stress is assumed to be constant through the thickness, so a shear correction factor is needed to account for the zero shear condition on the outer surface in an approximate manner.

With the wider applications of laminated composite beams and plates in engineering applications, much more attentions have been paid to the influence of shear deformation because fibrous composites and layered structures have generally low shear moduli. The constantly distributed shear stress assumption limited the applications of the Timoshenko beam theory to laminated composite beams. Thus, much efforts have to be made to develop refined or higher-order shear deformation theories. With different distribution of the transverse shear stress through the thickness, various refined theories have been proposed. Reddy [2] suggested a third-order shear deformation theory with a parabolic distribution of the shear stress, in which the number of unknowns kept the same as the first-order deformation theory [3], since the transverse deflection was constant through thickness. Touratier [4] added a sine term to the in-plane displacement, resulting in a cosine shear stress distribution through thickness. The other distributions of the shear stress are reported to be hyperbolic [5], exponential [6], amongst others [7, 8]. In the refined or higher-order shear deformation theories, the surface shear stresses vanish, avoiding the use of a shear correction factor. In recent years, with the aid of finite element formulation, a unified formulation of different high-order or refined beam theories has been proposed by Carrera et al. [9]. Comparison of different high-order theories has also been made by using the unified formulation [10]. For more details of beam theories, readers are referred to the comprehensive review articles by Ghugal and Shimpi [11], by Kulkarni et al. [12], and Shabanlou et al. [13], and the reference therein.

Another important engineering field, requiring higher-order theories, is the design of overtone resonators operating at high frequencies. Besides the first-order shear deformation theory for plates, which is essentially the counterpart of Timoshenko's beam theory in plates, Mindlin

[14] proposed a general theory for the high frequency vibrations of plates, starting with the expansion of all components of the displacement in power series. Following Mindlin's work, Lee and Nikodem [15] developed another general theory, starting with the expansion of all components of the displacement in a series of trigonometrical functions. All the above theories have been successfully applied to analyze vibration of plates, especially for quartz crystal plates vibrating at thickness-shear frequency. Although the above general theories might be referred to as higher-order theories, they may still required the shear correction factors in the approximations. The third-order and fifth-order overtone vibrations of quartz crystal plate were analyzed by Mindlin [16] and Wang et al. [17], respectively. Lee [18] improved his higher-order theory by adding an additional linear function of thickness coordinate in displacements, and using the assumption of parabolic distribution of the shear stress through thickness.

Since the infinite trigonometrical functions were the thickness modes of an infinite plate with top and bottom surfaces completely traction-free [19], Lee's plate theory was considered to be accurate for the analysis of natural vibration of plates at high frequencies when the thickness vibration modes are dominant. Beside, due to the usage of trigonometrical functions, it simplifies the application on the analysis of higher-order modes of beams. However, little work was reported on the analysis of vibration of beams by using Lee's methodology. The plate theory proposed by Lee [18] is rigorously reduced to a one-dimensional theory, i.e., an infinite system of one-dimensional equations, which is called Lee's beam theory (LBT) in this paper. The approximation of the infinite system of any order with proper truncation yields a corresponding higher-order beam theory. The first-order approximation of LBT is explored in detail in this paper, which leads to a novel first-order shear deformation theory of beams in flexural motion. In some sense, the present first-order shear deformation beam theory can be transformed into Timoshenko's beam theory (TBT). The numerical results of natural frequencies and mode shapes of beams with both ends free, clamped, and simply supported are presented for discussions of features and advantages of the proposed first-order shear deformation beam theory.

## 2. Plane stress problem

The theory begins with the two-dimensional elasticity problem. For homogenous and isotropic material, the relations between the stresses $T_{ij}$ and the strains $S_{ij}$ in the two-dimensional problem at the state of plane stress are given by Barber [20]

$$T_{ij} = \frac{2\mu}{1-\nu}\left[\nu\delta_{ij}S_{kk} + (1-\nu)S_{ij}\right], \quad i,j = 1,2 \tag{1}$$

where $\nu$ and $\mu$ are Poisson's ratio and the shear modulus, respectively. The relations between the strains $S_{ij}$ and the displacements $u_i$ are given by

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2. \tag{2}$$

The equations of motion are given by

$$T_{ij,i} = \rho\ddot{u}_j, \quad i,j = 1,2, \tag{3}$$

where the repeated subscript letter denotes the summary from 1 to 2.

According to Hamilton's principle, one has

$$\int_{t_0}^{t_1} dt \int_S \left(T_{ij,i} - \rho\ddot{u}_j\right)\delta u_j \mathrm{d}A = 0, \tag{4}$$

$$\int_{t_0}^{t_1} dt \oint \left(t_j - n_i T_{ij}\right)\delta u_j \mathrm{d}l = 0, \tag{5}$$

where $S$ is the body of the beam at the state of plane stress, $dl$ is the segment of boundary of the beam, $t_j$ is the traction on the boundary of beams and $n_i$ is the normal vector of the boundary.

## 3. A one-dimensional theory

In this section, a one dimensional theory is derived from the plane stress equations.

Following the methodology proposed by Lee [18], the displacements of plane stress problem can be assumed to be

$$u_i(x_1, x_2, t) = -bu_{2,i}^{(0)}\varphi + \sum_{n=0}^{\infty} u_i^{(n)}(x_1, t)\cos\frac{n\pi}{2}(1-\varphi), \quad i = 1,2 \tag{6}$$

where $\varphi = x_2/b$, $b$ is one-half of the thickness of the beam, see Fig. 1, and $u_i^{(n)}$ $(i = 1,2)$ are the $2n+2$ unknowns, also known as the displacements of order $n$, which are independent of $x_2$. The function $\varphi$ in the Fourier series form can be written as

$$\varphi = \sum_{n=0}^{\infty} c_n \cos\frac{n\pi}{2}(1-\varphi), \tag{7}$$

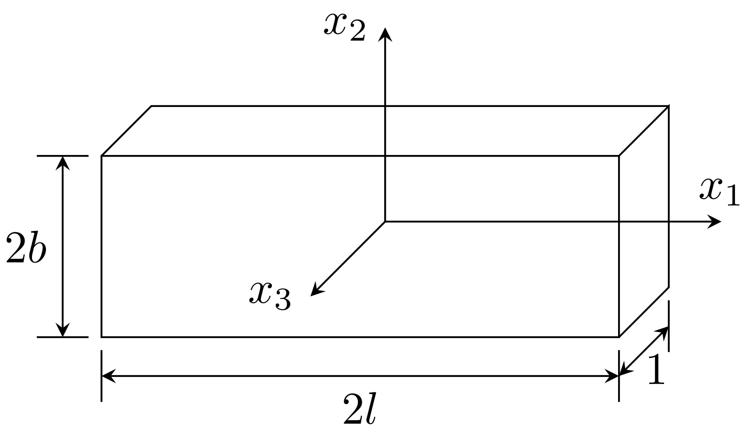


Figure 1: Geometry of a beam.

where the coefficients $c_n$ are given by

$$c_n = \int_{-1}^{1} \varphi \cos \frac{n\pi}{2}(1-\varphi)\mathrm{d}\varphi = \begin{cases} 8/(n^2\pi^2), & n = \text{odd}, \\ 0, & n = \text{even}. \end{cases} \tag{8}$$

It should be mentioned that the cosine functions in Eq. (6) are of physical meanings. They are simple thickness vibration modes of infinite plates [19]. Therefore, this one-dimensional theory takes the changes along thickness direction into account.

Substitution of Eq. (8) into the first term of the right-hand side of Eq. (6) leads to

$$u_i = \sum_{n=0}^{\infty} \left( u_i^{(n)} - bc_n u_{2,i}^{(0)} \right) \cos \frac{n\pi}{2}(1-\varphi), \quad i = 1, 2. \tag{9}$$

Then substituting Eq. (9) into (2) gives

$$S_{ij} = \sum_{n=0}^{\infty} \left[ S_{ij}^{(n)} \cos \frac{n\pi}{2}(1-\varphi) + \bar{S}_{ij}^{(n)} \sin \frac{n\pi}{2}(1-\varphi) \right], \tag{10}$$

where the strains of order $n$, $S_{ij}^{(n)}$ and $\bar{S}_{ij}^{(n)}$, are defined by

$$\begin{aligned} S_{ij}^{(n)} &= \frac{1}{2}\left( \delta_{1i} u_{j,1}^{(n)} + \delta_{1j} u_{i,1}^{(n)} \right) - bc_n u_{2,ij}^{(0)}, \\ \bar{S}_{ij}^{(n)} &= \frac{n\pi}{4b}\left[ \delta_{2i}\left( u_j^{(n)} - bc_n u_{2,j}^{(0)} \right) + \delta_{2j}\left( u_i^{(n)} - bc_n u_{2,i}^{(0)} \right) \right]. \end{aligned} \tag{11}$$

Thus the components of strains of order $n$, $S_{ij}^{(n)}$ and $\bar{S}_{ij}^{(n)}$, are given by

$$\begin{aligned} S_{11}^{(n)} &= u_{1,1}^{(n)} - bc_n u_{2,11}^{(0)}, & \bar{S}_{11}^{(n)} &= 0, \\ S_{22}^{(n)} &= 0, & \bar{S}_{22}^{(n)} &= \frac{n\pi}{2b} u_2^{(n)}, \\ S_{12}^{(n)} &= \frac{1}{2} u_{2,1}^{(n)}, & \bar{S}_{12}^{(n)} &= \frac{n\pi}{4b}\left( u_1^{(n)} - bc_n u_{2,1}^{(0)} \right). \end{aligned} \tag{12}$$

By substituting Eq. (9) into Eq. (4), and following the standard variational procedure of Hamilton principle, one obtains the $n$th-order stress equations of motion as follows:

$$T_{1j,1}^{(n)} - \frac{n\pi}{2b}\bar{T}_{2j}^{(n)} + \frac{1}{b}F_j^{(n)} = (1+\delta_{n0})\rho\left(\ddot{u}_j^{(n)} - bc_n\ddot{u}_{2,j}^{(0)}\right), \quad j = 1, 2, \tag{13}$$

where the stresses of order $n$, $T_{ij}^{(n)}$ and $\bar{T}_{ij}^{(n)}$, and the face-traction of order $n$, $F_j^{(n)}$, are defined as

$$\begin{aligned} T_{ij}^{(n)} &= \int_{-1}^{1} T_{ij}\cos\frac{n\pi}{2}(1-\varphi)\mathrm{d}\varphi, \\ \bar{T}_{ij}^{(n)} &= \int_{-1}^{1} T_{ij}\sin\frac{n\pi}{2}(1-\varphi)\mathrm{d}\varphi, \\ F_j^{(n)} &= T_{2j}(b) - (-1)^n T_{2j}(-b). \end{aligned} \tag{14}$$

Note that $\bar{T}_{ij}^{(0)} = 0$.

By substituting Eqs. (10) into Eq. (1) and further into Eq. (14), the relations between the stresses of order $n$ and the strains of order $n$ are obtained as

$$\begin{aligned} T_{ij}^{(n)} &= \frac{2\mu}{1-\nu}\left\{(1+\delta_{n0})\left[\nu\delta_{ij}S_{11}^{(n)} + (1-\nu)S_{ij}^{(n)}\right] + \sum_{m=0}^{\infty} B_{mn}\left[\nu\delta_{ij}\bar{S}_{22}^{(m)} + (1-\nu)\bar{S}_{ij}^{(m)}\right]\right\}, \\ \bar{T}_{ij}^{(n)} &= \frac{2\mu}{1-\nu}\left\{\nu\delta_{ij}\bar{S}_{22}^{(n)} + (1-\nu)\bar{S}_{ij}^{(n)} + \sum_{m=0}^{\infty} B_{nm}\left[\nu\delta_{ij}S_{11}^{(m)} + (1-\nu)S_{ij}^{(m)}\right]\right\}, \end{aligned} \tag{15}$$

where the relations $S_{22}^{(n)} = \bar{S}_{11}^{(n)} = 0$ are used, and

$$\begin{aligned} B_{mn} &= \int_{-1}^{1} \sin\frac{m\pi}{2}(1-\varphi)\cos\frac{n\pi}{2}(1-\varphi)\mathrm{d}\varphi, \\ &= \begin{cases} \frac{4m}{(m^2-n^2)\pi}, & m+n = \text{odd}, \\ 0, & m+n = \text{even}. \end{cases} \end{aligned} \tag{16}$$

The boundary conditions can be derived from Eq. (5) as

$$\begin{cases} T_{2j} = t_j \text{ or } u_j = \hat{u}_j, & x_2 = \pm b, \\ T_{1j}^{(n)} = t_j^{(n)} \text{ or } u_j^{(n)} - bc_n\delta_{1j}u_{2,1}^{(0)} = \hat{u}_j^{(n)} - bc_n\delta_{1j}\hat{u}_{2,1}^{(0)}, & x_1 = \pm l, \end{cases} \tag{17}$$

where $\hat{u}_j$ and $\hat{u}_j^{(n)}$ are the prescribed displacements, $t_j$ is the prescribed traction and $t_j^{(n)}$ is defined by

$$t_j^{(n)} = \int_{-1}^{1} t_j\cos\frac{n\pi}{2}(1-\varphi)\mathrm{d}\varphi. \tag{18}$$

So far, an infinite system of one-dimensional equations has been presented via Eqs. (11), (13) and (15). Since the system is one-dimensional, it is a general beam theory. This beam theory can give accurate results approaching the solutions of two-dimensional systems.

In the following, we focus on the first-order approximation of this general beam theory. Special attention is paid on the equations of flexural motion of beams.

## 4. The first-order approximation

### *4.1. The differential equations in the first-order approximation*

The first-order approximation begins from the assumptions that can be imposed on the displacements as follows

$$u_1^{(n)} = 0, \quad n > 1; \quad u_2^{(n)} = 0, \quad n > 2. \tag{19}$$

The retention of the displacement $u_2^{(2)}$ permits the inclusion of strain of order 2, $\bar{S}_{22}^{(2)}$, but the other higher-order strains and stresses are disregarded.

In the absence of traction on the top and bottom surfaces, the stress equations of motion, Eq. (13), is reduced to

$$\begin{aligned} T_{11,1}^{(0)} &= 2\rho\ddot{u}_1^{(0)}, \\ T_{12,1}^{(0)} &= 2\rho\ddot{u}_2^{(0)}, \\ T_{11,1}^{(1)} - \frac{\pi}{2b}\bar{T}_{21}^{(1)} &= \rho\left(\ddot{u}_1^{(1)} - bc_1\ddot{u}_{2,1}^{(0)}\right), \\ T_{12,1}^{(1)} - \frac{\pi}{2b}\bar{T}_{22}^{(1)} &= \rho\ddot{u}_2^{(1)}. \end{aligned} \tag{20}$$

A substitution of Eq. (12) to Eq. (15) leads to the stress-displacement equations, required by Eq. (20), as follows:

$$\begin{aligned} T_{11}^{(0)} &= \frac{4\mu}{1-\nu}\left(u_{1,1}^{(0)} + \frac{\nu}{b}u_2^{(1)}\right), \\ T_{12}^{(0)} &= 2\mu\left[u_{2,1}^{(0)} + \frac{1}{b}\left(u_1^{(1)} - \frac{8b}{\pi^2}u_{2,1}^{(0)}\right)\right], \\ T_{11}^{(1)} &= \frac{2\mu}{1-\nu}\left(u_{1,1}^{(1)} - \frac{8b}{\pi^2}u_{2,11}^{(0)} + \frac{8\nu}{3b}u_2^{(2)}\right), \\ \bar{T}_{21}^{(1)} &= \bar{T}_{12}^{(1)} = \frac{\pi\mu}{2b}u_1^{(1)}, \\ T_{12}^{(1)} &= \mu u_{2,1}^{(1)}, \\ \bar{T}_{22}^{(1)} &= \frac{2\mu}{1-\nu}\left(\frac{\pi}{2b}u_2^{(1)} + \frac{4\nu}{\pi}u_{1,1}^{(0)}\right). \end{aligned} \tag{21}$$

From Eqs. (20) and (21), it can be observed that the displacement of order $n$, $u_j^{(n)}$, when $j+n$ is odd, and the displacement of order $n$, $u_j^{(n)}$, when $j+n$ is even, are uncoupled with each other.

Eqs. $(20)_{2,3}$ and $(21)_{2,3,4}$, involving the displacements $u_j^{(n)}$ with $j+n$ even, contribute to the flexural motion or anti-symmetric mode of the beam. Eqs. $(20)_{1,4}$ and $(21)_{1,5,6}$, involving the displacements $u_j^{(n)}$ with $j+n$ odd, contribute to the extensional motion or symmetric mode of the beam. The boundary conditions are prescribed by Eq. $(17)_2$. Therefore, the first-order approximation of LBT can be used to analyze both the flexural and extensional motions of beams. Due to the uncoupling of the two above mentioned motions of isotropic beam, the flexural motion of beams is investigated in the following for an intuitive assessment of accuracy of LBT.

*4.2. The differential equations of flexural motion of beams*

Following Lee's methodology, the displacement $u_2^{(2)}$ is related to the displacements $u_2^{(0)}$ and $u_1^{(1)}$ by the equation,

$$\frac{8}{3b}u_2^{(2)} = -\nu\left(u_{1,1}^{(1)} - \frac{8b}{\pi^2}u_{2,11}^{(0)}\right), \tag{22}$$

which is derived from Eq. $(15)_1$ with assumption $T_{22}^{(1)} = 0$. To improve the accuracy of theory for flexural motions, the parabolic distribution of the shear deformation through the thickness direction is proposed. In the first-order approximation, the assumption leads to [18]

$$T_{12}^{(0)} = \frac{\pi^3}{24}\bar{T}_{12}^{(1)}. \tag{23}$$

By virtue of Eqs. $(21)_{3,4}$, (22) and (23), Eqs. $(20)_{2,3}$ yield the differential equations in terms of the displacements $u_2^{(0)}$ and $u_1^{(1)}$, i.e.,

$$\begin{gathered}\frac{\pi^4\mu}{96b}u_{1,1}^{(1)} = \rho\ddot{u}_2^{(0)},\\ E\left(u_{1,11}^{(1)} - \frac{8b}{\pi^2}u_{2,111}^{(0)}\right) - \frac{\pi^2\mu}{4b^2}u_1^{(1)} = \rho\left(\ddot{u}_1^{(1)} - \frac{8b}{\pi^2}\ddot{u}_{2,1}^{(0)}\right),\end{gathered} \tag{24}$$

where Young's module $E$ is given by $E = 2(1+\nu)\mu$. Eq. (24) are differential equations for flexural motion of beams.

The corresponding boundary conditions for the flexural motion are as follows:

For simply supported end

$$u_2^{(0)} = 0, \quad u_{1,1}^{(1)} - \frac{8b}{\pi^2}u_{2,11}^{(0)} = 0, \tag{25}$$

where the second equation is derived from $T_{11}^{(1)} = 0$;

For free end

$$u_1^{(1)} = 0, \quad u_{1,1}^{(1)} - \frac{8b}{\pi^2}u_{2,11}^{(0)} = 0, \tag{26}$$

which are derived from $T_{12}^{(0)} = 0$ and $T_{11}^{(1)} = 0$, respectively;

For clamped end

$$u_2^{(0)} = 0, \quad u_1^{(1)} - \frac{8b}{\pi^2} u_{2,1}^{(0)} = 0. \tag{27}$$

*4.3. The displacement and strain of flexural motion of beams*

The components of displacement in the first-order approximation of flexural motion are extracted from Eq.(6), that is

$$\begin{aligned} u_1 &= -b u_{2,1}^{(0)} \varphi + u_1^{(1)} \sin\frac{\pi\varphi}{2}, \\ u_2 &= u_2^{(0)} - u_2^{(2)} \cos\pi\varphi, \end{aligned} \tag{28}$$

which is a truncation of the series in Eq. (9) with $u_1^{(0)}$ and $u_2^{(1)}$ disregarded. Eq. (28) suggests that a cross-section of beam are not necessary a plane after the deformation.

The components of strain are given by

$$\begin{aligned} S_{11} &= -b u_{2,11}^{(0)} \varphi + u_{1,1}^{(1)} \sin\frac{\pi\varphi}{2}, \\ S_{22} &= \frac{\pi}{2} u_2^{(2)} \sin\pi\varphi, \\ S_{12} &= \frac{\pi}{4b} u_1^{(1)} \cos\frac{\pi\varphi}{2} - \frac{1}{2} u_{2,1}^{(2)} \cos\pi\varphi. \end{aligned} \tag{29}$$

From Eq. (29), the strain, as well as stress, changes along the thickness direction, while it is constant in Timoshenko's beam theory. On the top and bottom surfaces, the component of strain, $S_{22}$, vanishes, but the other components, $S_{11}$ and $S_{12}$, remain, which leads to the existence of traction, $T_{j2}$ $(j = 1, 2)$, on the top and bottom surfaces, quite similar with Timoshenko's beam theory. However, instead of shear correction factors, the assumption of parabolic distribution of the shear deformation through the thickness direction, Eq. (23), is considered in the present theory.

Like TBT, the first-order approximation of LBT for flexural motion of beams is a first-order shear deformation theory.

*4.4. Verification of flexural motion of beams in the first-order approximation*

*4.4.1. The relation with Timoshenko's beam theory*

In the following the first-order approximation of present beam theory for the analysis of flexural motion of beams is named by LBT1st for short. For a better understanding on the theory, it is reasonable to have a comparison between LBT1st and TBT for the analysis of flexural motion of beams, since both theories are first-order shear deformation theories. Higher-order approximations of LBT are required for comparisons with higher-order theories, for example, a third-order approximation of LBT is suggested for a comparison with Reddy's third-order shear deformation theory. It is not surprised to find that the displacements $u_2^{(0)}$ and $u_1^{(1)}$ in LBT1st can be transformed to the deflection $y$ and the bending slope $\psi$ in Timoshenko's beam theory. By using the transformation

$$y = \frac{8b}{\pi^2}u_2^{(0)}, \quad \psi = -u_1^{(1)} + \frac{8b}{\pi^2}u_{2,1}^{(0)}, \tag{30}$$

the differential equations, Eq. (24), and the boundary conditions, Eqs. (25)-(27), are identically transformed into those given by Timoshenko's theory of beams with the shear correction factor $\lambda = \pi^2/12$, which is the same as the shear correction factor in Mindlin's plate theory [3].

Although the differential equations and boundary conditions of the unknowns in LBT1st and TBT are identical, the displacements are not the same. Once the unknowns $u_1^{(1)}$ and $u_2^{(0)}$ are obtained, the components of displacement in TBT are given by

$$u_1 = b\varphi(u_1^{(1)} - \frac{8b}{\pi^2}u_{2,1}^{(0)}), \quad u_2 = \frac{8b}{\pi^2}u_2^{(0)}, \tag{31}$$

which is different from the displacement in LBT1st, i.e., Eq. (28).

The elimination of $u_1^{(1)}$ or $u_2^{(0)}$ from Eq. (24) leads to one-dimensional differential equations in terms of $u_2^{(0)}$ or $u_1^{(1)}$ as followings

$$\begin{aligned} EI\frac{\partial^4 u_2^{(0)}}{\partial x_1^4} + \rho A\frac{\partial^2 u_2^{(0)}}{\partial t^2} - \rho I\left(1 + \frac{12E}{\pi^2\mu}\right)\frac{\partial^4 u_2^{(0)}}{\partial x_1^2\partial t^2} + \frac{12\rho^2 I}{\pi^2\mu}\frac{\partial^4 u_2^{(0)}}{\partial t^4} = 0, \\ EI\frac{\partial^4 u_1^{(1)}}{\partial x_1^4} + \rho A\frac{\partial^2 u_1^{(1)}}{\partial t^2} - \rho I\left(1 + \frac{12E}{\pi^2\mu}\right)\frac{\partial^4 u_1^{(1)}}{\partial x_1^2\partial t^2} + \frac{12\rho^2 I}{\pi^2\mu}\frac{\partial^4 u_1^{(1)}}{\partial t^4} = 0, \end{aligned} \tag{32}$$

where $I = 2b^3/3$ is the second moment of area of the cross section and $A = 2b$ is the area of the cross section with the width of the beam as unity, see Fig. 1. Eq. (32) is the same as the differential equation in TBT with the shear correction factor $\lambda = \pi^2/12$.

By assuming the displacement $u_2^{(0)}$ as a straight-crested wave so that

$$u_2^{(0)} = A_2^{(0)}\sin(\xi x_1)\mathrm{e}^{\mathrm{i}\omega t}, \tag{33}$$

where $A_2^{(0)}$, $\xi$ and $\omega$ are the amplitude, wave number, and frequency, respectively, substitution of Eq. (33) into Eq. (32) yields the equation for the dispersion curves of the straight-crested wave given by

$$\Omega^4 - \left(\frac{E}{\mu} + \frac{\pi^2}{12}\right)Z^2\Omega^2 - \Omega^2 + \frac{\pi^2}{12}\frac{E}{\mu}Z^4 = 0, \tag{34}$$

where the relation $I = 2b^3/3$ is used and the dimensionless variables $\Omega$ and $Z$ are defined by

$$\Omega = \frac{2b\omega}{\pi}\sqrt{\frac{\rho}{\mu}}, \quad Z = \frac{2b\xi}{\pi}, \tag{35}$$

where $\frac{\pi}{2b}\sqrt{\frac{\mu}{\rho}}$ is usually regarded as the fundamental frequency of the thickness-shear modes.

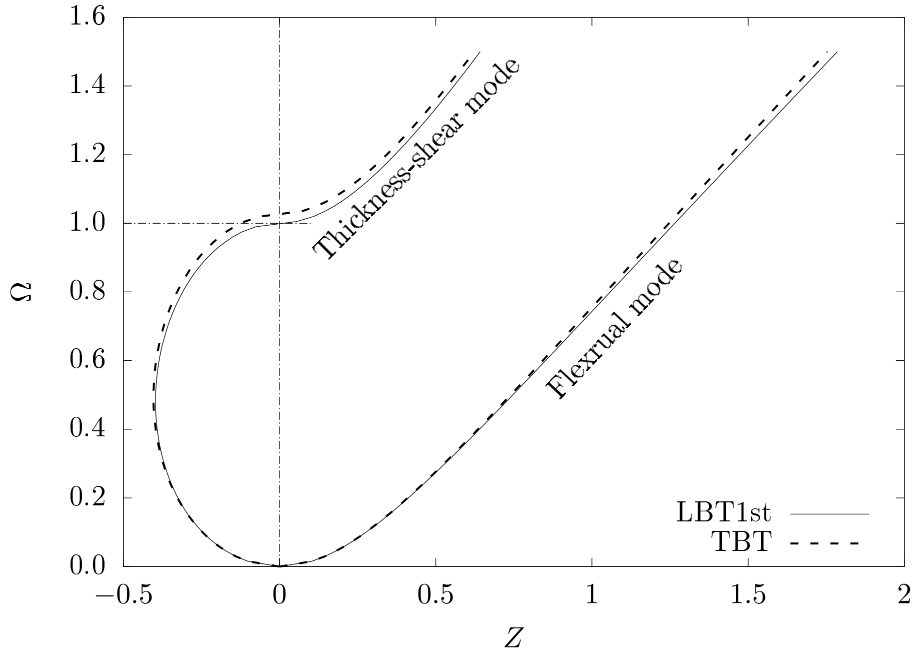


Figure 2: Dispersion curves of straight crested waves in a beam via LBT1st and TBT with correction factor $5(1+\nu)/(6+5\nu)$ and Poisson's ratio 0.3.

Fig. 2 shows the dispersion curves of beams via LBT1st, and TBT with correction factor $5(1+\nu)/(6+5\nu)$. The chosen correction factor was assumed to be accurate for the analysis of vibration of beams [21, 22]. The difference between solid and dashed lines in Fig. 2 terms from the shear correction factors. If the correction factor in TBT is $\pi^2/12$, then the dispersion curves by the two beam theories are identical. The cut-off frequency predicted by LBT1st is equal to the fundamental frequency of the thickness-shear modes, which is well-known in the analysis of vibration of plates. The cut-off frequency predicted by TBT with correction factor $5(1+\nu)/(6+5\nu)$ is higher than the fundamental frequency of the thickness-shear modes. Therefore, for an accurate cut-off frequency, LBT1st or TBT with correction factor $\pi^2/12$ is suggested.

### *4.4.2. Numerical examples and discussions*

In this section, regarding the two-dimensional plain stress theory as a reference, LBT1st is numerically compared with TBT. An analysis of vibrations of finite beams is presented.

Following previous analyses [21, 22], the beam illustrated by Fig. 1 has thickness $2b = 0.125$ m, unitary width (according to the plane stress theory), Young's modulus $\mathrm{E} = 210 \times 10^9$ N/m$^2$, density $\rho = 7850$ kg/m$^3$, and Poisson's ratio $\nu = 0.3$. The length of beam $2l$ changes in different examples. For simplify, both ends of a beam are simply supported, free, or clamped.

The analytical solutions of frequencies and modes of a finite beam via LBT1st can be found in Appendix A. Due to the symmetry of boundary conditions, the solutions in Appendix A include only the antisymmetric part of $u_2^{(0)}$ and symmetric part of $u_1^{(1)}$. The analytical solutions via TBT are similar [23]. The solutions via plane stress theory are given by a commercial finite element software.

Table 1 contains the first four dimensionless frequencies $\Omega$ of beams with both end simply supported via LBT1st, TBT, plane stress theory (PST). The number of $n$ in the table represent the $n$-th root of frequency equations in Appendix A. The ratios of length to thickness of beams are 2, 3 and 4, respectively. The difference of numerical results via the three theories is acceptable in engineering applications. In most cases, the frequencies predicted by TBT are a little more accurate than those by LBT1st, see Fig. 3. Similar situation holds when both ends of a beam are clamped, see Tab. 2 and Fig. 4. However, when both ends are free, the frequencies predicted by LBT1st are a little more accurate than those by TBT, see Tab. 3 and Fig. 5.

| | $l/b = 4$ | | | $l/b = 3$ | | | $l/b = 2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | LBT1st | TBT | PST | LBT1st | TBT | PST | LBT1st | TBT | PST |
| 1 | 0.27397 | 0.27636 | 0.27657 | 0.42481 | 0.42978 | 0.43038 | 0.74336 | 0.75530 | 0.75730 |
| 2 | 0.74336 | 0.75530 | 0.75731 | 1.06594 | 1.08612 | 1.08988 | 1.70373 | 1.74151 | 1.74786 |
| 3 | 1.22653 | 1.25104 | 1.25568 | 1.52991 | 1.55234 | 1.49540 | 1.96720 | 1.98744 | 1.88247 |
| 4 | 1.33437 | 1.35795 | 1.31495 | 1.70373 | 1.74151 | 1.74790 | 2.64147 | 2.70560 | 2.70957 |

Table 1: Dimensionless frequencies $\Omega$ of beams with both ends simply supported calculated by LBT1st, TBT, and PST, respectively.

The mode shapes of beams with different ratios of length to thickness at specific frequencies are presented in Figs. 6-8. The shapes calculated by PST are assumed to be exact. According to the figures, LBT1st predicts mode shapes of beams more accurately than TBT. Take the case when $n = 4$ and $l/b = 3$ in Fig. 6 for example, with the displacements of four corners of the beam predicted by the three theories almost the same, the mode shape predicted by LBT1st is

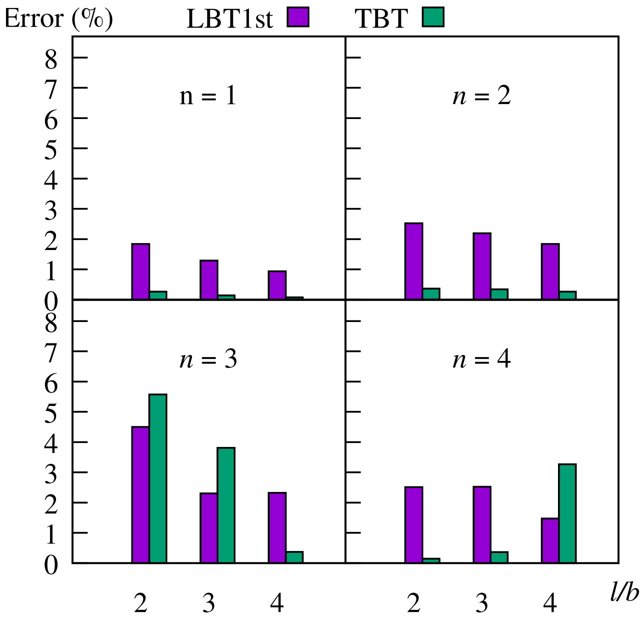


Figure 3: Percent relative errors of dimensionless frequencies to PST via LBT1st and TBT when both ends are simply supported; $n$ is the order of vibration mode, and $l/b$ is ratio of length to thickness.

| | $l/b=4$ | | | $l/b=3$ | | | $l/b=2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | LBT1st | TBT | PST | LBT1st | TBT | PST | LBT1st | TBT | PST |
| 1 | 0.33245 | 0.33465 | 0.33906 | 0.48147 | 0.48575 | 0.49368 | 0.77343 | 0.78444 | 0.79934 |
| 2 | 0.76297 | 0.77405 | 0.78333 | 1.05579 | 1.07822 | 1.08192 | 1.32606 | 1.35172 | 1.31608 |
| 3 | 1.09361 | 1.11967 | 1.09116 | 1.18512 | 1.20631 | 1.18644 | 1.71678 | 1.74962 | 1.77709 |
| 4 | 1.23357 | 1.25682 | 1.26896 | 1.69445 | 1.73443 | 1.74620 | 2.58456 | 2.66779 | 2.47020 |

Table 2: Dimensionless frequencies $\Omega$ of beams with both ends clamped calculated by LBT1st, TBT, and PST, respectively.

| | $l/b=4$ | | | $l/b=3$ | | | $l/b=2$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | LBT1st | TBT | PST | LBT1st | TBT | PST | LBT1st | TBT | PST |
| 1 | 0.38461 | 0.39315 | 0.38805 | 0.57122 | 0.58870 | 0.57789 | 0.86825 | 0.89869 | 0.87891 |
| 2 | 0.83345 | 0.85822 | 0.84615 | 1.02522 | 1.04676 | 1.03153 | 1.27293 | 1.29195 | 1.27148 |
| 3 | 1.06688 | 1.08801 | 1.06899 | 1.28697 | 1.32593 | 1.29466 | 1.87594 | 1.93251 | 1.84467 |
| 4 | 1.28217 | 1.31379 | 1.28785 | 1.59911 | 1.61236 | 1.57878 | 2.26018 | 2.26605 | 2.13654 |

Table 3: Dimensionless frequencies $\Omega$ of beams with both ends free calculated by LBT1st, TBT, and PST, respectively.

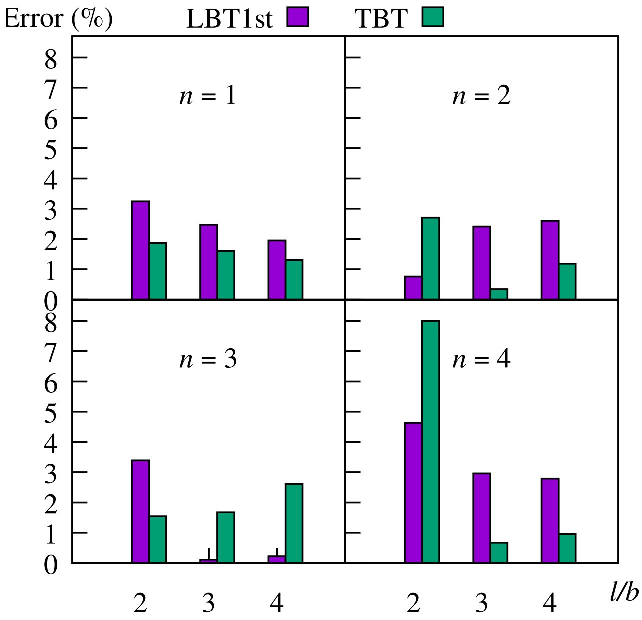


Figure 4: Percent relative errors of dimensionless frequencies to PST via LBT1st and TBT when both ends are clamped; $n$ is the order of vibration mode, and $l/b$ is ratio of length to thickness.

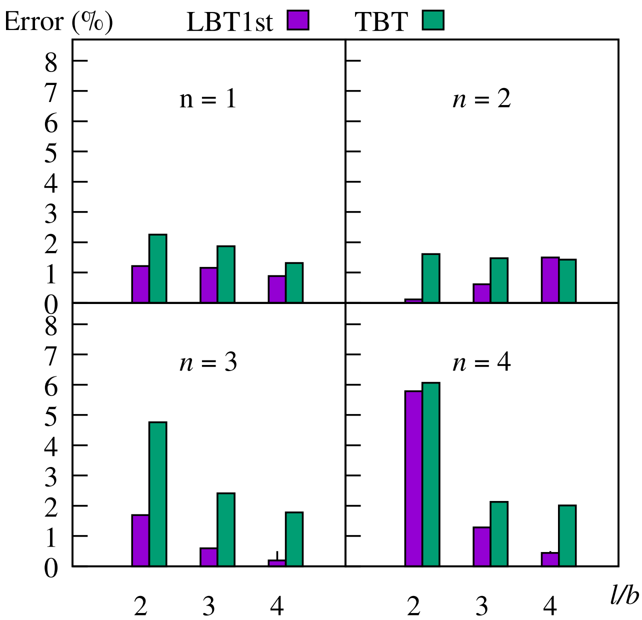


Figure 5: Percent relative errors of dimensionless frequencies to PST via LBT1st and TBT when both ends are free; $n$ is the order of vibration mode, and $l/b$ is ratio of length to thickness.

quite similar with that predicted by PST, while there is significant difference between the mode shapes predicted by TBT and PST. Similar cases can be found in Figs. 6 and 7. According to the mode shape predicted by PST, the verticle line (a cross-section of beams) is no longer straight during the vibration. Eq. (31) suggests that the verticle line in TBT keeps straight during the vibration, but Eq. (28) suggests that the verticle line in LBT1st is not necessary straight during the vibration. This is the reason why LBT1st predicted mode shapes better than TBT. In the case when verticle line keeps almost straight during the vibration, e.g., the case when $n = 1$ and $l/b = 4$ in Fig. 8, the difference between LBT1st and PST is not so significant. Therefore LBT1st might be an alternative to TBT on the analysis of vibration of beams with a little improvement on descriping the field displacements of beams.

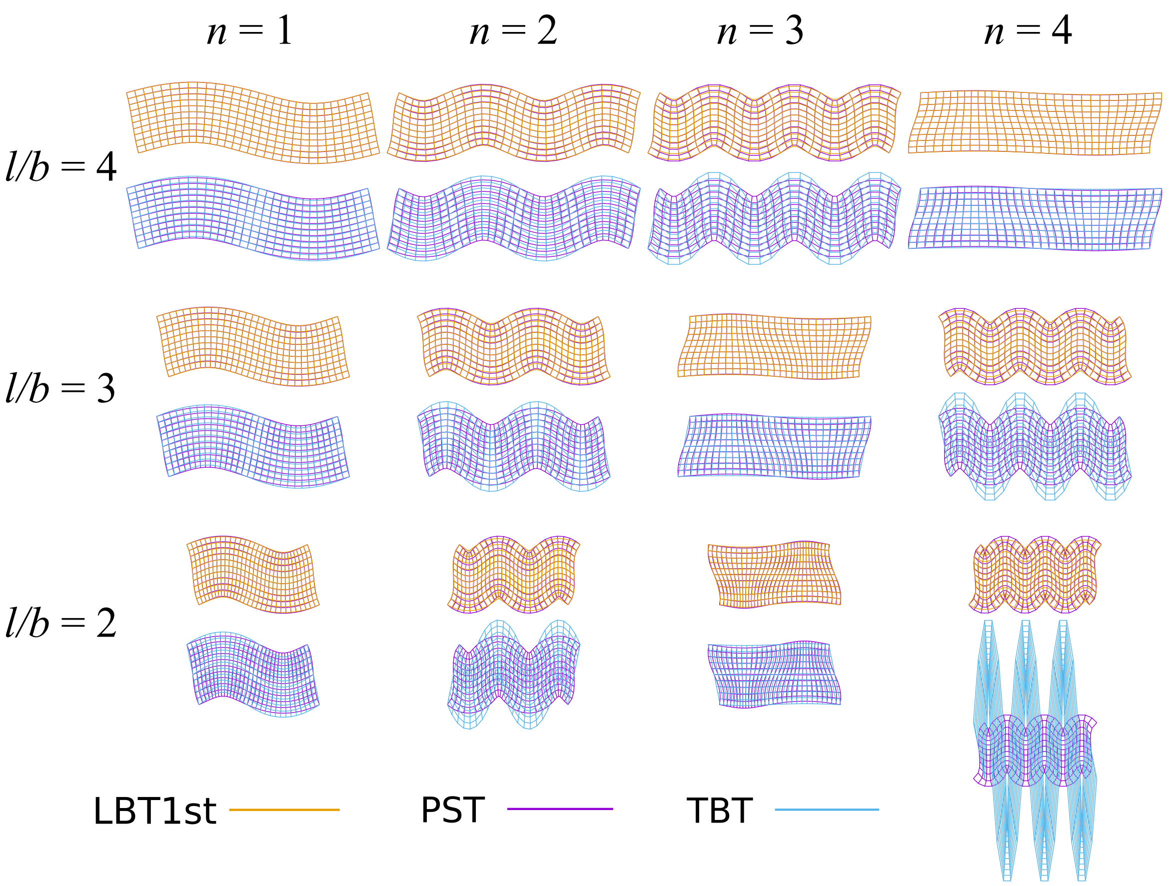


Figure 6: Mode shapes of beams for different order number $n$ and ratio of length to thickness $l/b$ when both ends are simply supported by using LBT1st, TBT, and PST, respectively.

## 5. Conclusions

Plate theory used in the literature to carry out free vibration analysis in the high frequency range proposed by Lee has been successfully deduced to a one-dimensional theory in an elegant,

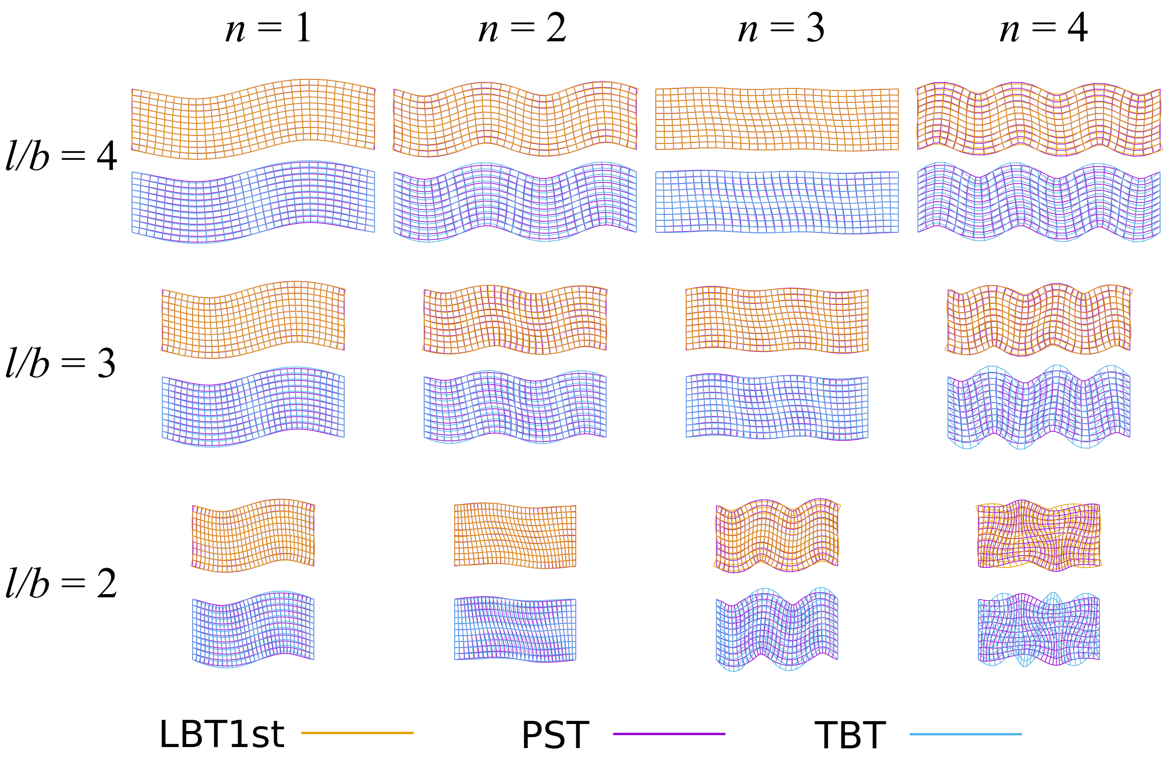


Figure 7: Mode shapes of beams for different order number $n$ and ratio of length to thickness $l/b$ when both ends are clamped by using LBT1st, TBT, and PST, respectively.

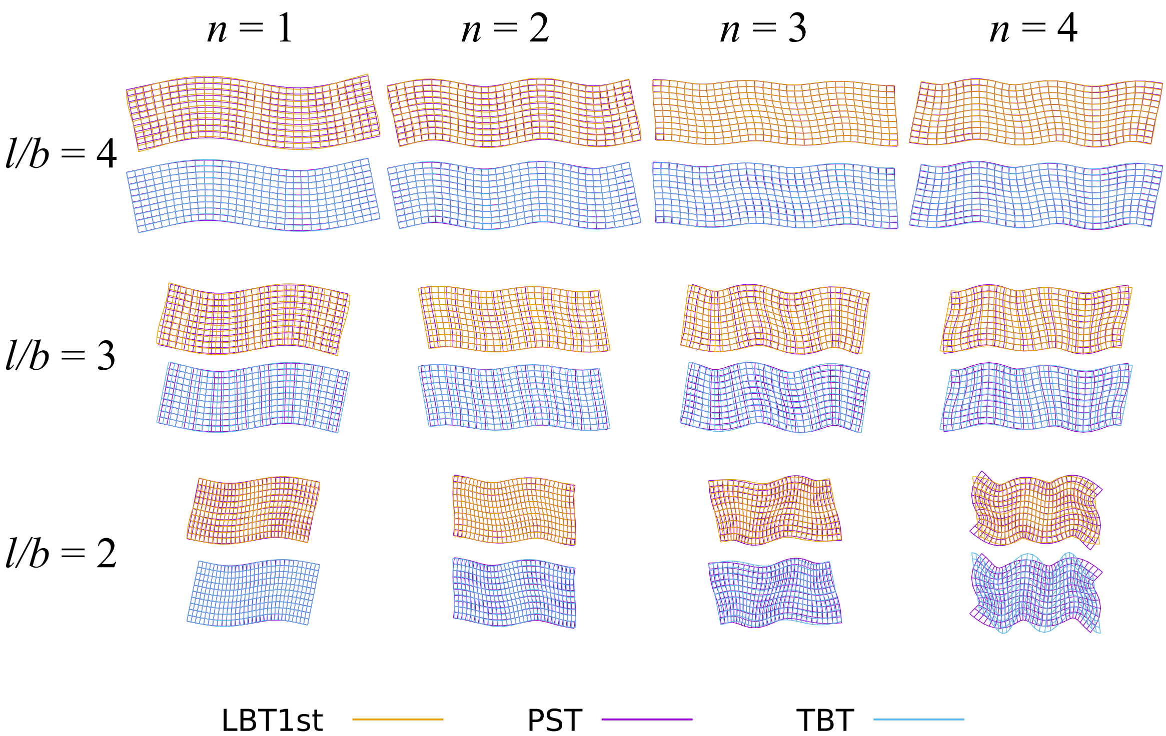


Figure 8: Mode shapes of beams for different order number $n$ and ratio of length to thickness $l/b$ when both ends are free by using LBT1st, TBT, and PST, respectively.

accurate, and computationally efficient manner. The infinite system of equations of motion presented by the theory can be used to analyze higher-order modes of vibrations of beams. The first-order approximation leads to a beam theory considering shear effect. By using an appropriate transformation, the differential equations and boundary conditions in LBT1st are transformed to those in TBT as a special case. The frequencies calculated by the two theories can be identical if the correction factor in TBT is $\pi^2/12$, while the displacements are different slightly. Unlike TBT, a cross-section of beam is not necessary a plane during the vibration in LBT1st. In general, LBT1st is a little better at simulating mode shapes than TBT.

## 6. Acknowledgments

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## Appendix A. Solutions of present beam theory

Now we consider a beam with both ends simply supported, free, or clamped. The length of the beam is $2l$, and the ends are at $x_1 = \pm l$.

By virtue of the symmetry of the boundary condition, the antisymmetric part of $u_2^{(0)}$ and symmetric part of $u_1^{(1)}$ are of interest.

From Eq. (34), the solutions of $Z$ are given by

$$Z_{1,2}^2 = \frac{1}{2}\left(\frac{12}{\pi^2} + \frac{\mu}{E}\right)\Omega^2 \pm \frac{1}{2}\sqrt{\left(\frac{12}{\pi^2} + \frac{\mu}{E}\right)^2 \Omega^4 + \frac{48\mu}{\pi^2 E}\Omega^2(1-\Omega^2)}, \tag{A.1}$$

The expressions of the general solutions of the antisymmetric part of $u_2^{(0)}$ and symmetric part of $u_1^{(1)}$ depends on the value of $\Omega$.

When $Z_2^2 > 0$, $\Omega > 1$,

$$\begin{aligned} u_2^{(0)} &= \alpha_1 C_1 \sin(\xi_1 x_1) e^{i\omega t} + \alpha_2 C_2 \sin(\xi_2 x_1) e^{i\omega t}, \\ u_1^{(1)} &= C_1 \cos(\xi_1 x_1) e^{i\omega t} + C_2 \cos(\xi_2 x_1) e^{i\omega t}, \end{aligned} \tag{A.2}$$

where,

$$\xi_i = \frac{\pi \bar{Z}_i}{2b}, \quad \alpha_i = \frac{\pi^3 \bar{Z}_i}{48\Omega^2}, \quad \bar{Z}_i = \sqrt{Z_i^2}, \quad i = 1, 2. \tag{A.3}$$

When $Z_2^2 \leqslant 0$, $\Omega \leqslant 1$,

$$\begin{aligned} u_2^{(0)} &= \alpha_1 C_1 \sin(\xi_1 x_1) e^{i\omega t} + \alpha_2 C_2 \sinh(\xi_2 x_1) e^{i\omega t}, \\ u_1^{(1)} &= C_1 \cos(\xi_1 x_1) e^{i\omega t} + C_2 \cosh(\xi_2 x_1) e^{i\omega t}, \end{aligned} \tag{A.4}$$

where $\xi_1$, $\xi_2$ and $\alpha_1$ are defined by Eq. (A.3), but $\alpha_2$ is defined by

$$\alpha_2 = -\frac{\pi^3 \bar{Z}_2}{48\Omega^2}, \quad \bar{Z}_2 = \sqrt{-Z_2^2}, \quad i = 1, 2. \tag{A.5}$$

As a special case, when $\Omega = 1$, one has

$$\begin{aligned} u_2^{(0)} &= \alpha_1 C_1 \sin(\xi_1 x_1) e^{i\omega t}, \\ u_1^{(1)} &= C_1 \cos(\xi_1 x_1) e^{i\omega t} + C_2. \end{aligned} \tag{A.6}$$

When both ends are simply supported, a substitution of Eqs. (A.2) and (A.4) into the boundary condition Eq. (25) at $x_1 = \pm l$ yields the equations of the unknowns $C_1$ and $C_2$. For the nontrivial solution of $C_1$ and $C_2$, the determinant of the coefficients should vanish. The final equations determining the frequencies are given by

$$\begin{cases} \sin(\xi_1 l) \sin(\xi_2 l) = 0, & \Omega > 1, \\ \sin(\xi_1 l) = 0, & \Omega \leqslant 1. \end{cases} \tag{A.7}$$

When both ends are free, the equations for determining the frequencies are given by

$$\begin{aligned} (\xi_2 - \frac{8b}{\pi^2}\alpha_2 \xi_2^2) \cos(\xi_1 l) \sin(\xi_2 l) - (\xi_1 - \frac{8b}{\pi^2}\alpha_1 \xi_1^2) \cos(\xi_2 l) \sin(\xi_1 l) = 0 \quad (\Omega > 1), \\ (\xi_2 - \frac{8b}{\pi^2}\alpha_2 \xi_2^2) \cos(\xi_1 l) \sinh(\xi_2 l) + (\xi_1 - \frac{8b}{\pi^2}\alpha_1 \xi_1^2) \cosh(\xi_2 l) \sin(\xi_1 l) = 0 \quad (\Omega \leqslant 1). \end{aligned} \tag{A.8}$$

When both ends are clamped, the equations for determining the frequencies are given by

$$\begin{aligned} \alpha_1(1 - \frac{8b}{\pi^2}\alpha_2 \xi_2) \cos(\xi_2 l) \sin(\xi_1 l) - \alpha_2(1 - \frac{8b}{\pi^2}\alpha_1 \xi_1) \cos(\xi_1 l) \sin(\xi_2 l) = 0 \quad (\Omega > 1), \\ \alpha_1(1 - \frac{8b}{\pi^2}\alpha_2 \xi_2) \cosh(\xi_2 l) \sin(\xi_1 l) - \alpha_2(1 - \frac{8b}{\pi^2}\alpha_1 \xi_1) \cos(\xi_1 l) \sinh(\xi_2 l) = 0 \quad (\Omega \leqslant 1). \end{aligned} \tag{A.9}$$

Once the normalized frequency $\Omega$ is determined, the unknowns $u_2^{(0)}$ and $u_1^{(1)}$ are given by Eq. (A.2) or Eq. (A.4). Once the ratio of $C_2/C_1$ is determined from Eq. $(24)_1$, the modes at specific frequencies are given by Eq. (28).

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